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Abstract: Given a finite set of non-collinear points in the plane there exists a line that passes through exactly two points. Such a line is called an *ordinary line*. An efficient algorithm for computing such a line was proposed by Mukhopadhyay et al [MAH97].

In this note we extend this result in two directions. We first show how to use this algorithm to compute an *ordinary conic*, that is, a conic passing through exactly five points, assuming that all the points do not lie on the same conic. Our proof of existence and the consequent algorithm is simpler than previous ones. We also show how to compute an ordinary hyperplane in three and higher dimensions.

Key-words: computational geometry.

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Recherche d'une conique ordinaire et d'un hyperplan ordinaire

Résumé :

Étant donné un ensemble de points non alignés dans le plan, il existe une droite passant par exactement deux de ces points. Une telle droite est appelée *droite ordinaire*. Une méthode efficace de détermination d'une telle droite a été proposée par Mukhopadhyay et al.

Nous étendons ce travail dans deux directions. Nous utilisons cet algorithme pour déterminer une *conique ordinaire*, c'est-à-dire une conique passant par exactement cinq points, si les points ne sont pas tous sur une même conique. Notre preuve d'existence et notre algorithme sont sensiblement plus simple que les travaux précédents. Nous montrons également comment déterminer une *hyperplan ordinaire* en trois dimensions et plus.

Mots-clés : géométrie algorithmique.

1 Introduction

Let $\mathcal{S}$ be a set of n points in the plane. A *connecting* line of $\mathcal{S}$ is a line that passes through at least two of its points. A connecting line is said to be *ordinary* if it passes through exactly two points of $\mathcal{S}$.

The problem of establishing the existence of such a line originated with Sylvester [Syl93], who proposed the following problem in 1893:

> If n points in the plane are such that a line passing through any two of them passes through a third point, then are the points collinear?

No solution came forth during the next forty years. In 1943, a positive version of the same problem was proposed by [Erd44], and in the following year a solution by Gallai [Gal44] appeared in print.

Subsequently other proofs also appeared, notable among which were the proofs by Steinberg [Ste44] and Kelly [KM58]. These results showed that the answer is in the affirmative for real projective geometry in the plane. Therefore if the points of $\mathcal{S}$ are not collinear then there is at least one ordinary line. In fact, [KM58] showed that there are at least $3n/7$ ordinary lines.

A set of points is said to be *co-conic* if all the points lie on one conic. In this paper we address a more general version of the ordinary line problem: given a set of points that are not co-conic, find a conic that passes through exactly five points.

Our algorithm provides a constructive proof of the existence of such a conic. Another proof is contained in [WW88]. Our proof is very simple and allows us to relate a result on the number of ordinary lines to the number of ordinary conics.

The paper is organised as follows. In the next section we discuss some mathematical preliminaries. The algorithm is discussed in the third section. We conclude in the fourth and final section.

2 Preliminaries

2.1 Notations

Let $\mathcal{S}$ be a set of n points in $\mathbb{R}^2$. Let ϕ be the transformation that maps a point $p = (x, y) \in \mathbb{R}^2$ to the point $p^\star = (x^2, y^2, xy, x, y) \in \mathbb{R}^5$. $\mathbb{R}^{2\star}$ is the 2 dimensional manifold image of $\mathbb{R}^2$ in $\mathbb{R}^5$ by this transformation; let $\mathcal{S}^\star = \phi(\mathcal{S})$ be the image of $\mathcal{S}$ under this transformation.

If $\mathcal{C}$ is a conic in $\mathbb{R}^2$ described by the equation $ax^2 + by^2 + cxy + dx + ey + f = 0$, then $\phi(\mathcal{C}) = \mathcal{C}^\star$ is the intersection of $\phi(\mathbb{R}^2) = \mathbb{R}^{2\star}$ with the hyperplane $\mathcal{C}^v$: $au + bv + cw + dx + ey + f = 0$ in $\mathbb{R}^5$.

If A and B are two subsets of $\mathbb{R}^5$, we define the affine hull $A \oplus B$ as the set of linear combination of points in A and B.

$$A \oplus B = \{q \in \mathbb{R}^5; \exists \alpha, \beta \in \mathbb{R},\ \alpha + \beta = 1,\ \exists a \in A\ \exists b \in B \quad q = \alpha a + \beta b\}$$

2.2 Ordinary line

For completeness, we sketch briefly the algorithm for finding an ordinary line in a set of coplanar points.

Let l be a directed line (direction $\vec{v}$) through exactly one point p_0 of $\mathcal{S}$. Let $q_\lambda = p_0 + \lambda\vec{v}$. We find the line defined by points of $\mathcal{S}$ that cut l in a point q_λ with minimal $\lambda > 0$. Such a line passes through two points consecutive in polar order around p_0 and can thus be found in $O(n \log n)$ time. Either this line is ordinary or a line through p_0 and a point on this line is ordinary. For details see [MAH97].

3 Algorithm

The idea behind the algorithm is to find a hyperplane that passes through exactly five points of $\mathcal{S}^\star$. In the $\mathbb{R}^2$ plane this corresponds to a conic that passes through exactly five points of $\mathcal{S}$.

We first find a conic that passes through exactly three points of $\mathcal{S}$. We do this as follows. We choose $p, q, r \in \mathcal{S}$ and $s, t \notin \mathcal{S}$ such that no three (four) of the five points are collinear. Denote by $\vec{\imath}$ the vector $(1, 0) \in \mathbb{R}^2$ and consider the conic $\mathcal{A}_\theta$ passing through the five points $p, q, r, s, t + \theta\vec{\imath}$.

For any point $\rho \in \mathcal{S}$, there exist at most two values $\theta = \theta_\rho$ or $\theta = \theta'_\rho$ such that $\rho \in \mathcal{A}_\theta$. This is because if

$$a_\theta x^2 + b_\theta y^2 + c_\theta xy + d_\theta x + e_\theta y + f_\theta = 0$$

is the conic that passes through the points, $p, q, r, s, t + \theta\vec{\imath}$ then each of the coefficients is of second degree in θ. So it is easy to determine some θ_0 different from all these values such that $\mathcal{A}_{\theta_0} \cap \mathcal{S} = \{p, q, r\}$.

Now the affine hull $\mathcal{B} = p^\star \oplus q^\star \oplus r^\star$ is a subspace of the hyperplane $\mathcal{A}^\star_{\theta_0}$ and so is the affine hull spanned by the points $p^\star$, $s^\star$ and $(t + \theta_0\vec{\imath})^\star$. Moreover, these subspaces have disjoint directions.

Let γ be a point of $\mathbb{R}^5$ not in $\mathcal{A}^\star_{\theta_0}$. Translate the affine hull of the points, $p^\star$, $s^\star$ and $(t+\theta\vec{\imath})^\star$ to pass through γ. Let $\mathcal{C}$ denote this translated affine hull.

For any point $\rho \in \mathcal{S} - \{p,q,r\}$, we construct the point $\rho^\dagger = (\mathcal{B} \oplus \rho^\star) \cap \mathcal{C}$. The intersection is exactly one point because suitable translates of the flats $\mathcal{B} \oplus \rho^\star$ and $\mathcal{C}$ are supplementary subspaces of the vectorial space $\mathbb{R}^5$. Otherwise, $\rho^\star$ would belong $\mathcal{A}^\star_{\theta_0}$ - impossible since $\rho \notin \mathcal{A}_{\theta_0}$ by the definition of θ_0.

The set of points $\mathcal{S}^\dagger = \left\{\rho^\dagger; \rho \in \mathcal{S} \ - \{p,q,r\}\right\}$ lies in the two-dimensional plane $\mathcal{C}$. Let l be a line in that plane and $\mathcal{H}$ the hyperplane through $\mathcal{B}$ and l. By construction $\rho^\dagger \in \mathcal{H}$ if and only if $\rho^\star \in \mathcal{H}$. Thus at this point we see a complete equivalence between the problem of finding an ordinary line for $\mathcal{S}^\dagger$ and an ordinary conic for $\mathcal{S}$ through the points p, q and r.

If all the points of $\mathcal{S}^\dagger$ are collinear, then all the points of $\mathcal{S}^\star$ are in the same hyperplane and hence all the points of $\mathcal{S}$ are co-conic.

Otherwise, there exists an ordinary line l in $\mathcal{C}$, and the corresponding hyperplane $\mathcal{H}$ contains only five points of $\mathcal{S}^\star$. The corresponding conic is ordinary (and pass through p,q,r).

The following theorem is a consequence of the above discussion and the fact that there are at least $\frac{3n}{7}$ ordinary lines:

> **Theorem:** *Given $\mathcal{S}$ a set of points not co-conic in the plane, then for any three non colinear points in $\mathcal{S}$, there exists at least $\frac{3(n-3)}{7}$ ordinary conics of $\mathcal{S}$ passing though these three points. An ordinary conic can be found in $O(n \log n)$ time.*

4 Ordinary plane in three dimensions

Given a set S of n points in three space, a connecting plane (that is, a plane through some three points of S) is defined to be ordinary if it has all but one of its points on a line. A plane that passes through exactly three points is certainly ordinary in the sense of this definition but such a plane need not exist. As an example, place three or more points on each of two skew lines in three space. This configuration of points has no connecting plane that is defined by exactly three points [Mot51].

We show that the ideas sketched in Section 2.2 can be generalized to three and higher dimensions to compute a plane that is ordinary in the sense of the above definition.

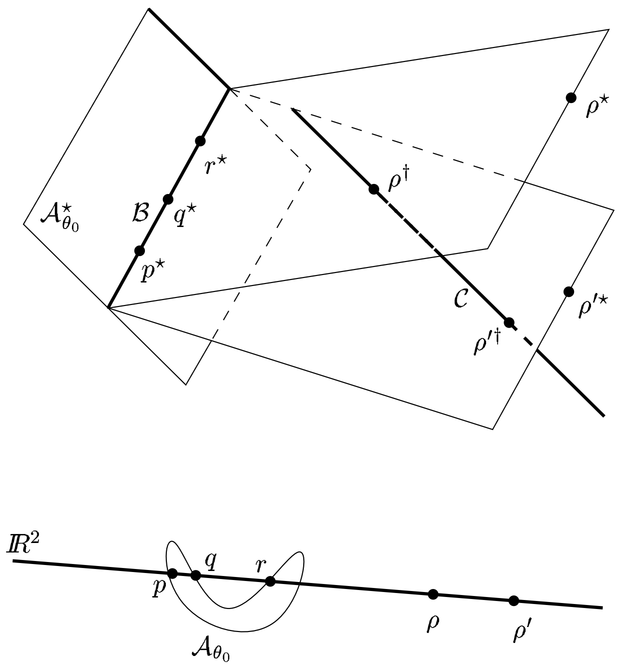

Figure 1: Mapping in 5 dimensions

Let p_0 be a point of $\mathcal{S}$ and γ a line through p_0. Let Λ be a plane through three points p_1, p_2, p_3 of $\mathcal{S}$ such that its distance to p_0, measured along γ, is minimum among all possible connecting planes of $\mathcal{S}$ that intersect γ.

If Λ is ordinary we are done; otherwise, set $g = \gamma \cap \Lambda$ and let Γ be an arbitrary plane containing γ; finally, set $\lambda = \Lambda \cap \Gamma$.

Let p_1, p_2, p_3 and p_4 be points of $\mathcal{S}$ in Λ such that no 3-tuple of the form $p_1 p_i p_j$, $i, j \in \{2, 3, 4\}$, are collinear (such points exist in Λ since Λ is not ordinary). We consider the planes through $p_0 p_1$ and p_2, p_3, p_4 respectively. Let L_2, L_3 and L_4 be their respective intersections with Γ and l_2, l_3 and l_4 their respective intersections with λ.

Assume, without loss of generality, that l_2 is separated from g along λ by l_3 or l_4. Then we claim that the plane determined by the points p_0, p_1, p_2 is ordinary. That is all the points, barring p_0, are on the line determined by p_1, p_2. If not, let q be a point, distinct from p_0, lying outside this line.

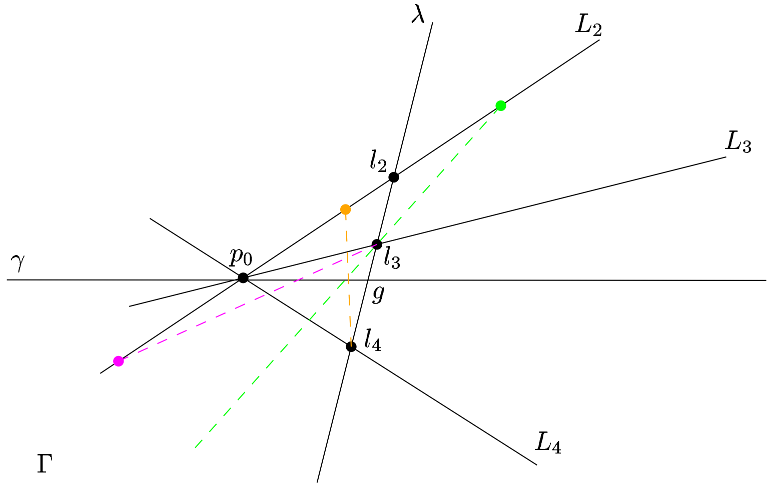

Figure 2: Ordinary plane in three dimensions

In fact, if $q \notin \Lambda \cap L_2$, one of the two planes qp_1p_3 or qp_1p_4 must pass between g and p_0, contradicting the definition of Λ (see Figure 2).

It remains to find Λ efficiently. It is clear that if we consider the cell in the arrangement of the $O(n^3)$ planes defined by points of $\mathcal{S} - \{p_0\}$, that contains p_0, Λ is incident on the facet that is hit by γ.

By a point-plane duality transformation, with p_0 as the centre of inversion, the cell containing p_0 is mapped into the convex hull of the $O(n^3)$ vertices of an arrangement of $n-1$ planes.

In two dimensions we have to compute the convex hull of the vertices of an arrangement of $n-1$ lines and it is not difficult to see that a vertex can be on the convex hull only if the two lines have consecutive slopes (in the set of all slopes). In three dimensions, the phenomenon is similar if we consider the Gaussian diagram of the normals to the $n-1$ planes (the convex hull of the unit normal vectors to $n-1$ planes). Three planes define a vertex of the convex hull only if their normal vectors define a face of the Gaussian diagram. Since the Gaussian diagram can be computed in $O(n \log n)$ time, we get the following result:

> **Theorem:** *If $\mathcal{S}$ a set of n non-coplanar points in 3-space, then an ordinary plane can be found in $O(n \log n)$ time.*

The same ideas extend to higher dimensions. We can find an ordinary hyperplane with the help of a Gaussian diagram. The complexity is identical to the complexity of computing the convex hull in that dimension.

5 Conclusions

In this note we have shown that an algorithm for finding a line through exactly two points of a given set of non-collinear points can be used to find a conic through exactly five points, if we assume that all the points are not co-conic.

It is particularly easy to find an ordinary circle passing through a chosen point, p, of the given set of points, if we allow for a degenerate circle. We simply apply an inversion transformation with p as the centre of inversion. Solve the ordinary line problem for the remaining $n-1$ transformed points. We have a degenerate circle if the ordinary line found passes through p, else its image is an ordinary circle passing through p. We also conclude that at least $3(n-1)/7$ ordinary circles pass through a chosen point.

By applying a stereographic projection we note that if n points on a real sphere do not lie in the same plane then there is plane containing exactly three of them.

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